June 1964

SIMULATION OF A SEQUENTIAL NONPARAMETRIC PROCEDURE*

Richard Leroy Greenstreet

Technical Report No. 43

*This research was supported in part by the National Science Foundation under Grant Number G-19126.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to express gratitude to Professor Saul Blumenthal who guided me to completion of this thesis. Also, to Miss Kathy Smith who typed the manuscript so diligently.

Introduction

This paper presents a monte-carlo study of a sequential non-parametric test proposed by Blumenthal [5].

Chapter I describes the mechanics of the test and includes a computer program which performs the test. Chapter II describes the results of a computer simulation of the test. Small sample properties of the test are investigated. A comparison with a similar test by Lehmann is made also in chapter II. Chapter III presents the simulator program and interesting characteristics of the simulator are discussed.

CHAPTER I

Mechanics of Test

The purpose of this section is to describe the mechanics of the Sequential Sample Spacings test proposed by Blumenthal.

The test is one concerned with the null hypothesis that two parent populations are identical against general alternatives, i.e., H_0: $F = G$ (vs.) H_1: $F \neq G$.

The test's stopping rule is determined by two functions a_n and r_n, and a random variable d_n. a_n and r_n are defined by

1.1 $$a_n = \frac{\log \frac{\beta}{1-\alpha} + n \log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}},$$

1.2 $$r_n = \frac{\log \frac{1-\beta}{\alpha} + n \log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}},$$

where α and β are type I and type II error probabilities respectively, n is the sampling stage, p_0 is taken to be one-half $(p_0 = \frac{1}{2})$, and p_1 is a constant determined by the experimenter. (Two procedures for determining p_1 will be given later.) The random variable d_n is approximately binomial with parameters $(n, p = \frac{1}{2})$ under H_0, and $(n, p > \frac{1}{2})$ under H_1. The following rule is exercised at each sampling stage beginning with $n = 1$: accept H_0 if $d_n \leq a_n$, reject H_0 if $d_n \geq r_n$, and continue sampling if $a_n < d_n < r_n$. This test will terminate with probability one at some finite sample size.

If d_n is exactly binomial the above test is the Wald Sequential Probability Ratio test for testing a binomial mean with error probabilities α and β. In particular, H_0: $p = p_0$ (vs.) H_1: $p = p_1 > p_0$ [1].

The random variable d_n is related to the sample spacings at each stage, i.e., the number of X's falling between successive Y's in a combined ordered sample of X's from F and Y's from G. The nature of this relationship will be made apparent in the formal construction of d_n to be given now.

Consider random variables X and Y corresponding to F and G respectively. At the end of sampling stage n-1, having observed $X_1, X_2, \ldots, X_{3(n-1)}$; $Y_1, \ldots, Y_n$ independent observations from F and G, assume $a_{n-1} < d_{n-1} < r_{n-1}$. At the beginning of stage n observe three more X's, i.e., $X_{3n-2}, X_{3n-1}, X_{3n}$. Now put the samples $X_1, \ldots, X_{3n}$; $Y_1, \ldots, Y_n$ each in increasing order. Label the ordered observations $X_{n,1}, X_{n,2}, \ldots, X_{n,3n}$; $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}$. Form $S_1(n), \ldots, S_{n+1}(n)$, where $S_1(n)$ is the number of X's less than or equal to the smallest Y, $S_2(n)$ is the number of X's greater than the smallest Y, but less than or equal to the second smallest Y, etc., down to $S_{n+1}(n)$, the number of X's greater than the largest Y. (The $S_j(n)$ $j = 1, \ldots, n+1$ give the number of X's falling between successive Y's and are a measure of the distance between successive Y's.) Now put in increasing order the $S_j(n)$

$j = 1,\ldots,n + 1$. Label the ordered $S_j(n)$ as $S_{n,i}$ $i = 1,\ldots,n + 1$, then $S_{n,1} \leqq S_{n,2} \leqq \ldots \leqq S_{n,n+1}$. In case $S_k(n) = S_{k'}(n)$ $k < k'$, the ordering is $S_{n,1} \leqq S_{n,2} \leqq \ldots S_{n,k} \leqq S_{n,k'} \leqq \ldots S_{n,n+1}$.

Consider now the sum $\frac{1}{4n+1} \sum_{i=1}^{J_n} (S_{n,i} + 1)$, where J_n is defined as the integer such that $\frac{1}{4n+1} \sum_{i=1}^{J_n} (S_{n,i} + 1) \leqq \frac{1}{2} < \frac{1}{4n+1} \sum_{i=1}^{J_n+1} (S_{n,i} + 1)$.

Recall that the $S_{n,i}$ $i = 1,\ldots,n + 1$ are just the ordered $S_j(n)$ $j=1,\ldots,n+1$. Each $S_{n,i}$ $i = 1,\ldots,J_n$ going into the above sum corresponds to an $S_j(n)$, $j = k_1,\ldots,k_{J_n}$ $1 \leqq k_i \leqq n + 1$. These $S_j(n)$ in turn are directly related to intervals of the form $(Y_{n,j-1}, Y_{n,j}]$, where $j = k_1,k_2,\ldots,k_{J_n}$, and $Y_{n,0} = -\infty$, $Y_{n,n+1} = +\infty$. Call the collection of intervals of the above form I_n. Observe the next Y, i.e., Y_{n+1}. Define a random variable c_n in terms of Y_{n+1} and I_n as follows:

$$c_n = \begin{cases} 1 \text{ if } Y_{n+1} \in I_n \\ 0 \text{ if } Y_{n+1} \notin I_n \end{cases} .$$

The c_j's are approximately independent Bernoulli variates with $p = \frac{1}{2}$ under H_0, $p > \frac{1}{2}$ under H_1. Finally the random variable d_n is given by

1.3 $$d_n = \sum_{j=1}^{n} c_j .$$ d_n is the number of stages in which Y_{j+1} has fallen in the collection of intervals I_j for $j = 1,\ldots,n$.

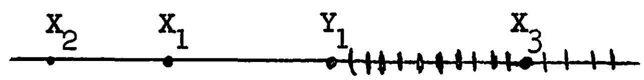
Accept H_0 if $d_n \leqq a_n$, reject H_0 if $d_n \geqq r_n$, continue sampling if $a_n < d_n < r_n$; i.e., observe $X_{3n+1}, X_{3n+2}, X_{3n+3}$ and repeat process.

The following example will make the above procedure more clear.

Let $n = 1$. <u>Stage I</u>.

Observe $X_1, X_2, X_3; Y_1$. Suppose ordering of X's gives

$X_2 < X_1 < Y_1 < X_3$. On the real line we have



, so

$S_1(1) = 2$, $S_2(1) = 1$. Then $S_{1,1} = 1$, $S_{1,2} = 2$.

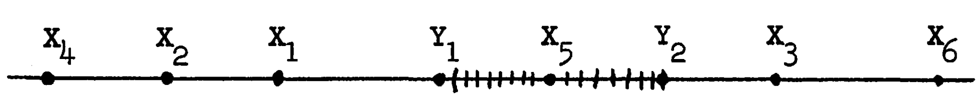
$$\frac{1}{4n+1}\sum_{i=1}^{J_n}(S_{n,i}+1) = \frac{1}{5}\sum_{i=1}^{J_n}(S_{n,i}+1)\ . \quad J_n = 1, \text{ since}$$

$$\frac{1}{5}\sum_{i=1}^{1}(S_{n,i}+1) = \frac{2}{5} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{5}\sum_{i=1}^{2}(S_{n,i}+1) = 1 > \frac{1}{2}\ .$$

Since $S_{1,1}$ corresponds to $S_2(1)$ the critical region I_1 is (Y_1, ∞).

Now observe Y_2. If $Y_2 \in I_1$, $c_1 = 1$ and $d_1 = 1$. If $Y_2 \notin I_1$, $c_1 = 0$ and $d_1 = 0$. Now compare d_1 with r_1 and a_1. Assume $a_1 < d_1 < r_1$. We go to <u>Stage II</u>.

For $n = 2$, observe X_4, X_5, X_6. Order $X_1, X_2, X_3, X_4, X_5, X_6; Y_1, Y_2$. Suppose the ordering gives



Then $S_1(2) = 3$, $S_2(2) = 1$, $S_3(2) = 2$, and $S_{2,1} = 1$, $S_{2,2} = 2$, $S_{2,3} = 3$.

$$\frac{1}{4n+1}\sum_{i=1}^{J_n}(S_{n,i}+1) = \frac{1}{9}\sum_{i=1}^{J_n}(S_{n,i}+1) .$$

$J_n = 1$, since $\frac{1}{9}\sum_{i=1}^{1}(S_{n,i}+1) = \frac{2}{9} \leq \frac{1}{2}$

and $\frac{1}{9}\sum_{i=1}^{2}(S_{n,i}+1) = \frac{5}{9} > \frac{1}{2}$.

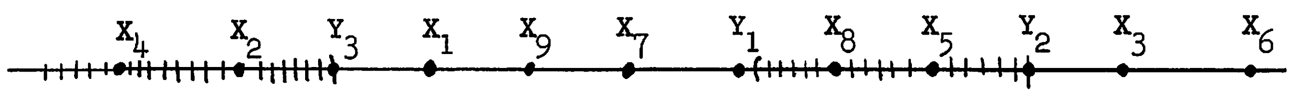
$S_{2,1}$ corresponds to $S_2(2)$. Therefore; I_2 is $(Y_1, Y_2]$. Now observe Y_3 .

$$c_2 = \begin{cases} 1 & Y_3 \in I_2 \\ 0 & Y_3 \notin I_2 \end{cases} , \quad d_2 = \sum_{i=1}^{2} c_i .$$

Now compare d_2 with r_2 and a_2. Let's assume $a_2 < d_2 < r_2$ and go on to the next stage. <u>Stage 3</u>.

For $n = 3$, observe X_7, X_8, X_9. Order $X_1, X_2, \ldots, X_8, X_9$; Y_1, Y_2, Y_3.

Suppose this ordering gives



Then $S_1(3) = 2$, $S_2(3) = 3$, $S_3(3) = 2$, $S_4(3) = 2$, and $S_{3,1} = 2$, $S_{3,2} = 2$, $S_{3,3} = 2$, $S_{3,4} = 3$.

$$\frac{1}{4n+1}\sum_{i=1}^{J_n}(S_{n,i}+1) = \frac{1}{13}\sum_{i=1}^{J_n}(S_{n,i}+1) .$$

Since $\frac{1}{13}\sum_{i=1}^{2}(S_{n,i}+1) = \frac{6}{13} \leq \frac{1}{2}$

and $\frac{1}{13}\sum_{i=1}^{3}(S_{n,i}+1) = \frac{9}{13} > \frac{1}{2}$,

$J_n = 2$. I_3 is $(-\infty\ Y_3] \cup (Y_1\ \ Y_2]$. Note: since

$S_{3,1} = S_{3,2} = S_{3,3} = 2$ and these correspond to $S_1(3) = S_3(3)$

$= S_4(3) = 2$, it would appear that I_3 could be formed in three ways,

namely, $(-\infty\ Y_3] \cup (Y_1, Y_2]$

or $(-\infty, Y_3] \cup (Y_2\ \infty]$

or $(Y_1, Y_2] \cup (Y_2\ \infty]$.

The convention in this case is to choose intervals from left to right, i.e., I_3: $(-\infty\ Y_3] \cup (Y_1, Y_2]$.

<u>Two Procedures for Determining p_1</u>.

I. The first procedure for determining p_1 gives $p_1 = p_1(N_0)$ where N_0 is a desirable expected sample size under H_0.

The simulation indicates that the expected sample size for Wald's S.P.R.T. [1] under H_0, given by

$$E_{H_0}(n) = \frac{(1-\alpha)\log\frac{\beta}{1-\alpha} + \alpha\log\frac{1-\beta}{\alpha}}{p_0\log\frac{p_1}{p_0} + (1-p_0)\log\frac{1-p_1}{1-p_0}},$$

tends to overestimate the average sample size of this test. (At least when F and G are normal variates.)

A p_1 may be determined by setting $E_{H_0}(n) = N_0$. This p_1 gives the Sequential Sample Spacings test an expected sample size under H_0 which is less than or equal to N_0.

Since $p_0 = \frac{1}{2}$, $E_{H_0}(n) = \dfrac{(1-\alpha)\log\frac{\beta}{1-\alpha} + \alpha\log\frac{1-\beta}{\alpha}}{\frac{1}{2}\log 4p_1(1-p_1)}$.

Setting $E_{H_0}(n) = N_0$, yields p_1.

II. p_1 may be determined as a function of distributions F_1 and G_1, against which power = $1-\beta$ is desired. $p_1 = p_1(F_1,G_1)$. For any F and G such that $p_1 = p_1(F,G) \geqq p_1(F_1,G_1) > \frac{1}{2}$ this test has power $\geqq 1-\beta$.

p_1 is determined in the above manner as follows: Let the appropriate F_1 and G_1 be chosen. Let f_1 and g_1 be density functions of F_1 and G_1 respectively. Consider

$$H(J_0,f_1,g_1) = 1 - \frac{3^{(J_0+1)}}{4}\int_{-\infty}^{\infty}\frac{[3 + (J_0+2)\frac{g_1(x)}{f_1(x)}]}{[3 + \frac{g_1(x)}{f_1(x)}]^{(J_0+1)}} f_1(x)dx \ .$$

J_0 is defined by

$$H(J_0,f_1,g_1) \leqq \frac{1}{2} < H(J_0+1,\ f_1,g_1) \ .$$

With J_0 now determined compute

$$A = \frac{4[\frac{1}{2} - H(J_0,f_1,g_1)]}{J_0+2} , \qquad \text{also}$$

$$P(J_0,f_1,g_1) = (J_0+2)3^{(J_0+1)}\int_{-\infty}^{\infty}\frac{g_1^{\,3}(x)}{f_1^{\,3}(x)[3 + \frac{g_1(x)}{f_1(x)}]^{(J_0+3)}} f_1(x)\, dx$$

and

$$P^*(J_0,f_1,g_1) = 1-3^{(J_0+1)}\int_{-\infty}^{\infty} \frac{3\left(\frac{g_1(x)}{f_1(x)}\right) + (J_0+2)\,\frac{g_1^{2}(x)}{f_1^{2}(x)}}{\left[3+\frac{g_1(x)}{f_1(x)}\right]^{(J_0+2)}}\,dx$$

and

$$Q(J_0,f_1,g_1) = 3^{(J_0+1)}\int_{-\infty}^{\infty} \frac{g_1^{2}(x)}{f_1^{2}(x)}\,\frac{f_1(x)}{\left[3+\frac{g_1(x)}{f_1(x)}\right]^{(J_0+2)}}\,dx \,.$$

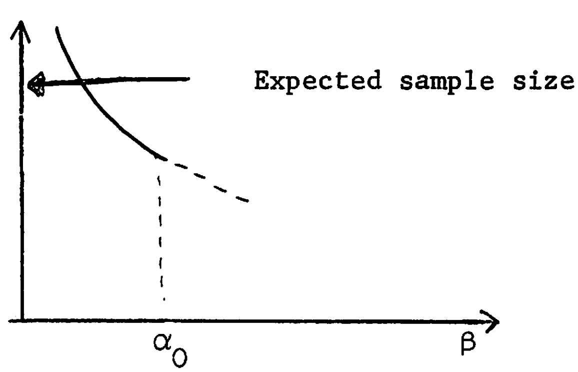
p_1 is now given by

$$p_1 = p^*(J_0,f_1,g_1) + \frac{A\ .\ P(J_0,f_1,g_1)}{Q(J_0,f_1,g_1)} \,.$$

If the integrals involved do not have analytic solutions, numerical techniques of solution are available which give sufficient accuracy.

Choosing β

The choice of β is up to the experimenter. However, if there is no criterion for choosing β in a particular manner, $\beta = \alpha$ is a reasonable choice. The figure below shows $\beta = \alpha$ minimizes the expected sample size as a function of β for all $\beta \leqq \alpha$.



Applying the Test

An application of the Sequential Sample Spacings test requires considerable computation. This section of the thesis presents an efficient computer program which handles the mechanics and computations of the test. The program is written in Fortran and Fortran Symbolic programming language for use on a Control Data 1604 computer.

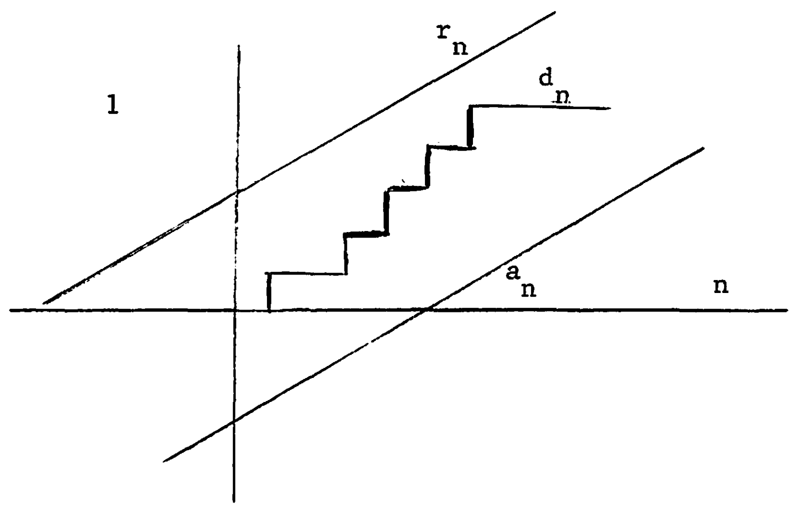
To use the program the experimenter must generate independent observations from F and G. The numerical values of these observations are punched onto data cards which are read into the computer under a specified format. The program performs the sequential test and prints out the decision at termination.

Recall that for $n = 1$, X_1, X_2, X_3 and Y_1, Y_2 are observed and the appropriate statistics computed. If $a_1 < d_1 < r_1$, X_4, X_5, X_6 and Y_3 are observed. (a_1, d_1, r_1 are defined by 1.1, 1.3 , 1.2 .) If $a_{n-1} < d_{n-1} < r_{n-1}$, that is, the decision is to continue ~~sampling,~~ X_{3n-2}, X_{3n-1}, X_{3n} and Y_{n+1} are observed during sampling stage n. A natural manner then in which to apply the sequential program is to observe X_1, X_2, X_3; Y_1, Y_2; read these observations into the computer which performs the test and makes a decision. If sampling continues, read X_4, X_5, X_6; Y_3 into the computer. Continue taking four observations at a time until termination.

Taking four observations at a time, however, is not the most efficient manner in which to apply the sequential program. A more efficient technique of sampling considers the functions determining the decision bounds, namely a_n and r_n, and the minimum number of samples starting at stage n necessary to reach termination.

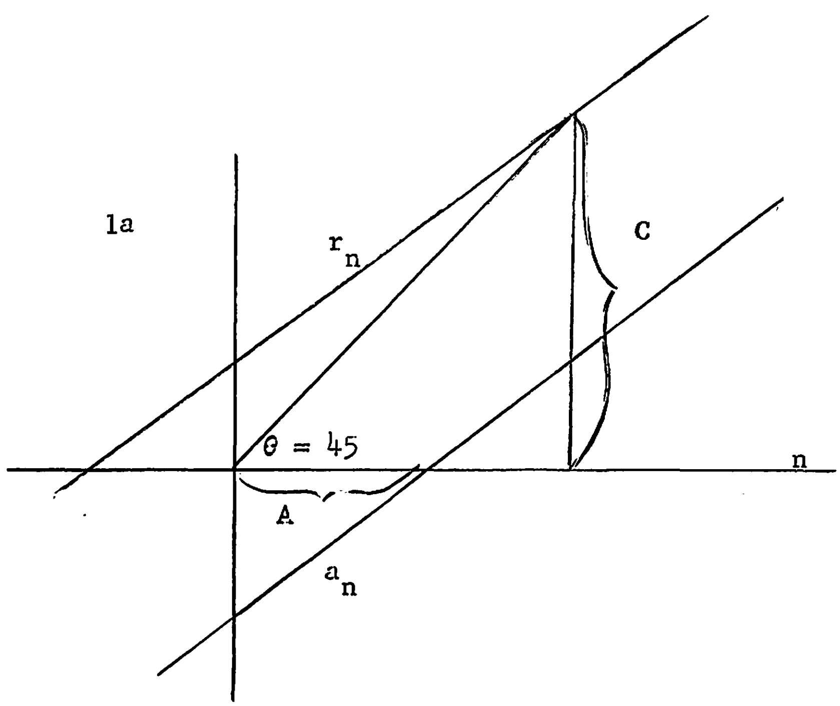
Figure 1 shows typical paths followed by the functions a_n and r_n and the random variable d_n.

1



Observe figure 1a, and see that the sequential test

1a



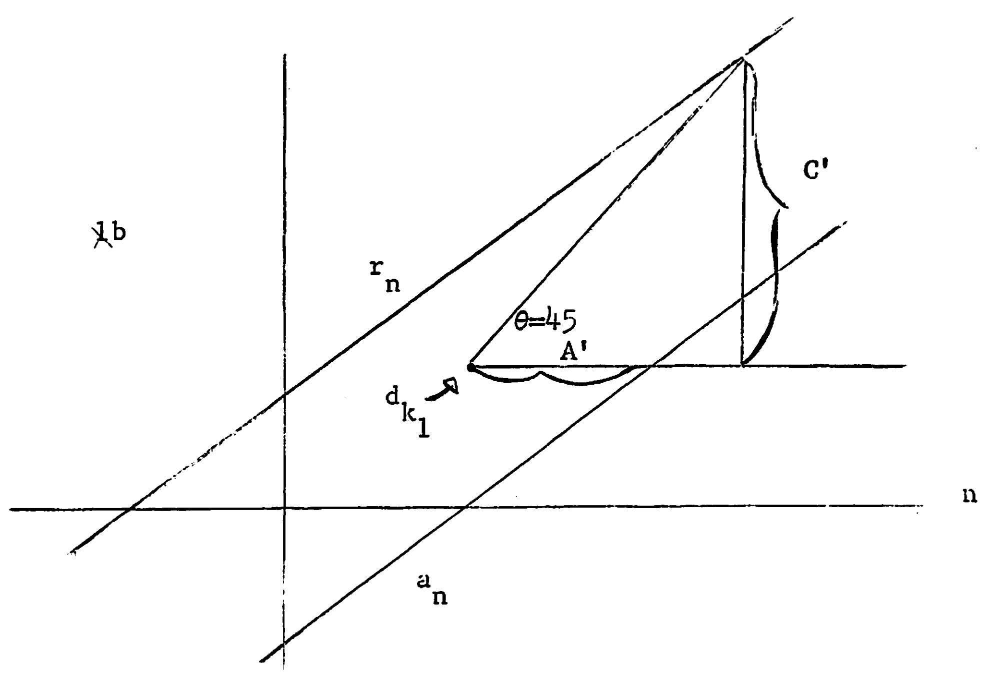
cannot terminate in less than $[\min(A,C)] + 1$ stages, where $[z]$ denotes the greatest integer less than or equal to z.

Solving the equations $a_n = 0$ and $r_n = n$ gives the numbers A and C which in general are not integers. At least $k_1 = [\min(A,C)] + 1$ stages must be carried through before termination is possible. This means $3k_1$ X's and $(k_1 + 1)$ Y's may be observed at the onset of the experiment without altering the properties of the sequential test.

The above technique may be repeated if the decision at stage k_1 is to continue sampling. Figure 1b shows the situation.

Note: If $k_1 > 1$, the sequential test must take the observations in their original order, i.e., X_1, X_2, X_3 and Y_1, Y_2 for $n = 1$, X_4, X_5, X_6 and Y_3 for $n = 2$ etc. The Sequential Sample Spacings test is not, in general, invariant under a permutation of the observations. In particular, the random variable d_n depends upon the order in which the observations are taken.

1b



Solving $a_n = d_{k_1}$ and $r_n = n - k_1 + d_{k_1}$, yields numbers n_1 and n_2 such that $A' = n_1 - k_1$ and $C' = n_2 - d_{k_1}$. Let $k_2 = [\min(A',C')] + 1$. At least k_2 additional stages must be completed before termination is possible. Again $3k_2$ X's and $(k_2 + 1)$ Y's may be observed without loss of sequential properties. This technique may be repeated until a decision to terminate is made.

To actually run the program included in the next few pages choose α and p_1, and set $A = \alpha$ and $PPP = p_1$, where A and PPP are parameters of the program. The program takes $\beta = \alpha$. Fixing A, sets $\alpha = \beta = A$. Set K3K and K1K2 equal to the final indices of X and Y to be read in respectively. For example, if X_1,X_2,X_3,X_4,X_5,X_6 and Y_1,Y_2,Y_3 are to be read in, K3K must equal 6 and K1K2 equal 3. If $X_7,X_8,X_9,X_{10},X_{11},X_{12}$ and Y_4,Y_5 are read, K3K = 12 and K1K2 = 5, etc.

All observations taken at one time from the experiment by the techniques described may be read into the computer at one time.

The computer stops at the completion of each stage of the sequential test. Simply push the "go" button to cycle through another stage.

Continue feeding the computer information until a decision to terminate is made by the program. At that time one of the statements "ACCEPT HNOT" or "REJECT HNOT" is printed.

Program for Applying S.S.S. Test

```
      DIMENSIONX(6000),  Y(2001),  S(2001), IC(2001), Z(2001), P(200),
     1  NTEMP(257)  ,  X1(6000) ,  R(3),  RI(3) ,  R2(1)
      COMMON X, Y, IC, Z, P, NTEMP, X1, R, RI, R2   ,S
      A=
      PPP=
      DD1=(LOGF(A/(1.-A)))/LOGF(PPP/(1.-PPP))
      DDD1=(LOGF(1./(2.*(1.-PPP))))/LOGF(PPP/(1.-PPP))
      EEE1=(LOGF(1./(2.*(1.-PPP))))/LOGF(PPP/(1.-PPP))
      EE1=(LOGF((1.-A  )/A  ))/LOGF(PPP/(1.-PPP))
      K1=1
      K3=1
      DO 10  N=1, 2000
   78 READ 77, K3K, K1K2
      READ  77, (X(J), J=K3, K3K), (Y(K), K=K1, K1K2)
   77 FORMAT(     )
      K1= K1K2+1
      K3=K3K+1
      B=N
      NN26= N+1
      NN1=N-1
      NN=3*N
      MN3=NN-3
      B4=4.*B+1 .
      N2=N+1
      IF(N-1)  73, 73, 74
```

```
73   CONTINUE
80   FORMAT(10E12.4, /)
     CALL NSORT(Y, Z, IC, N, 0, +6, NTEMP)
     CALL NSORT(X, X1 , IC, NN, 0, +6, NTEMP)
     GO TO 3
74   DO 75 K=1,  3
75   R(K)=X(MN3+K)
     CALL NSORT(R, RI , IC, 3, 0, -6, NTEMP)
     NXZ=0
     MN2=NN-2
     MN1=NN-1
     LDA(R+1)   LIL2(MN3).
     THS2(X+1)   SLJ0(84).
     SIL2(NXZ)    SLJ0(85).
 95  LDA(R+1)  STA2(X+2).
     LDA(R+2)   INI2(1).
     THS2(X+1)  SLJ0(86).
     SIL2(NXZ)   SLJ0(96)     .
 89  LDA(R+2)   STA2(X+2).
     LDA(R+3)   INI2(1)   .
     THS2(X+1)   SLJ0(99)    .
     SIL2(NXZ)   SLJ0(93) .
 98  LDA(R+3)   STA2(X+2).
     GO TO 82
 93  NNL=MN1-NXZ-1
     DO 94 I=1,  NNL
```

```
      KI=MN1-I+1
   94 X(KI+1)= X(KI)
      GO TO 98
   96 NNL=MN2-NXZ-1
      DO 88 I=1, NNL
      KI=MN2-I+1
   88 X(KI+1)= X(KI)
      GO TO 89
   85 IF(NXZ-MN3+1) 508, 95, 508
  508 NNL=MN3-NXZ-1
      DO 83 I=1, NNL
      KI=MN3-I+1
   83 X(KI+1)=X(KI)
      GO TO 95
   84 DO 81 K=1, MN3
      KJ=MN3-K+1
   81 X(KJ+3)= X(KJ)
      X(3)=R(1)
      X(2)=R(2)
      X(1)=R(3)
      GO TO 82
   86 DO 87 K=1, MN2
      KJ=MN2-K+1
   87 X(KJ+2)= X(KJ)
      X(2)=R(2)
```

```
      X(1)=R(3)
      GO TO 82
   99 DO 92 K=1, MN1
      KJ=MN1-K+1
   92 X(KJ+1)= X(KJ)
      X(1)=R(3)
   82  CONTINUE
      LDA(R2+1)  LIL2(NN1).
      THS2(Y+1)   SLJ0(500).
      SIL2(NXZ)   SLJ0(502).
  504 LDA(R2+1)   STA2(Y+2).
      GO TO 505
  500 DO 501  K=1, NN1
      KJ=NN1-K+1
 501  Y(KJ+1)=Y(KJ)
      Y(1)=R2(1)
      GO TO 505
  502 IF(NXZ-NN1+1)  509,504,509
  509 NNL=NN1-NXZ-1
      DO 503 I=1, NNL
      KI=NN1-I+1
 503  Y(KI+1)=Y(KI)
      GO TO 504
 505   CONTINUE
   3  I1=1
      DO 20 J=1,N
```

```
      JK=N-J+1
      SKOUNT=0.
      DO 18 I=I1, NN
      IK=NN-I+1
      IF(X(IK)-Y(JK))5, 5, 19
    5 SKOUNT=SKOUNT+1.
      IF (I-NN) 18, 24, 24
   18 CONTINUE
   19 S(J)=SKOUNT
      I1=I
   20 CONTINUE
      IR=3*N-I+1
      R=IR
      S(N+1)=R
      GO TO 26
   24 S(J)=SKOUNT
      KK= J+1
      N1= N+1
      JJ=J
      AJJ= JJ
      DO 25 K=KK, N1
   25 S(K)=0.
   26 NI= N+1
  910 CALL NSORT(S, Z, IC, NI, -1, -6, NTEMP)
      TEST=0.
      DO 30 J=1, 2001
```

```
      TEST=TEST+(S(J)+1.)/(B4)
      IF(TEST-.5) 30, 30, 29
   30 CONTINUE
   29 JJ=J-1
      BE=B*EEE1
      E1=EE1+BE
      D1=DD1+BE
      IF(N-1) 90,90,16
   90 EVENT=0.
  16  CONTUNUE
      R2(1)=Y(N+1)
      YN= Y(N+1)
      DO 40 I=1, JJ
      ICC=IC(I)
      ICK=N-ICC+1
       ICC1=ICC-1
      ICC1K=ICK+1
      IF(ICC-NI)  46, 47, 46
   47 IF(YN-Y(1))  40, 40, 57
   46 CONTINUE
      IF(ICC1)  56, 58, 56
   58 IF( YN-Y(N))  57, 57, 40
   56 CONTINUE
      IF(YN-Y(ICK))  55, 55, 40
   55 IF(YN-Y(ICC1K))  40, 40, 57
   40 CONTINUE
```

```
      GO TO 59
   57 EVENT=EVENT+1.
   59 CONTINUE
  912 CONTINUE
   63 IF(EVENT-D1)  60, 60, 62
   60 PRINT  70
   70 FORMAT( 1X,  36H ACCEPT HNOT
      PRINT  80,   EVENT, B, D1, E1
      CALL TIME(1HP, 1H1 )
      GO TO 11
   62 IF(EVENT-E1)  300, 64, 64
   64 PRINT  72
   72 FORMAT(1X,  36H REJECT HNOT
      PRINT  80, EVENT, B, D1, E1
      CALL TIME(1HP, 1H1 )
      GO TO 11
 300  CONTINUE
      STOP
   10 CONTINUE
   11 CONTINUE
   91 FORMAT (10E12.5)
 522  CONTINUE
      END
      END
```

CHAPTER II

The Simulation

A simulation of the Sequential Sample Spacings test was performed on the University of Minnesota's Control Data 1604 computer. Samples were taken from known populations in an experiment to study the properties of the test for usual values of the error probabilities. Small sample behavior for usual values of α was compared with behavior predicted when α approaches zero.

The average sample size of the Sequential Sample Spacings test at termination was determined when testing H_0: $F = G$ (vs) H_1: $F = F_1$, $G = G_1$ when sampling with H_0 true and with H_1 true, i.e., with $F = G$ and $F = F_1$, $G = G_1$. Varying values of α, the type I error probability, were used in each case; in particular $\alpha = .05, .01, .001, .0001$ and $.00001$.

The distributions from which samples were taken for the simulation were normal. With H_0 true, F and G were $N(0,1)$. Under H_1, for most experiments, F_1 was $N(0,1)$ and G_1 was $N(1,1)$. A few experiments were run with F_1: $N(0,1)$ and G_1: $N(1/2,1)$, with F_1: $N(0,1)$ and G_1: $N(1/4,1)$, and with F_1: $N(0,1)$, G_1: $N(1/8,1)$. α and β were chosen equal for the simulation. The following discussion is based primarily on the alternative F_1: $N(0,1)$, G_1: $N(1,1)$.

A convergence property of the Sequential Sample Spacings test proved by Blumenthal and investigated by the author's simulation is stated as follows: the limit of the ratio of the expected sample size for the Sequential Sample Spacings test, to the expected sample size for the Wald

Sequential Probability Ratio test equals one, as max (α,β) tends to zero.

The above property is stated formally as follows: Let $E_{H_i^*}(n)$ be the expected sample size under H_i^* for the Sequential Sample Spacings test, where H_i^* may be either H_0^*: $F = G$ or H_1^*: $F = F_1$, $G = G_1$. Similarly, let $E_{H_i}(n)$ be the expected sample size under H_i for the Wald Sequential Probability Ratio test, where H_i may be either H_0: $p = p_0 = \frac{1}{2}$ or H_1: $p = p_1 > \frac{1}{2}$. ($p_1 = p_1(F_1,G_1)$ is determined by the technique described in chapter I.) Let $\max(\alpha,\beta)$ be the maximum value of the error probabilities. Then, for $\epsilon_i > 0$, there exists $\delta_i > 0$ such that

$$\left| \frac{E_{H_i^*}(n)}{E_{H_i}(n)} - 1 \right| < \epsilon_i$$

whenever $\max(\alpha,\beta) < \delta_i$, $i = 0,1$.

$E_{H_0}(n)$ and $E_{H_1}(n)$ for the Wald test ([1] page 100) are given by 2.1 and 2.2 respectively.

2.1 $$E_{H_0}(n) = \frac{(1-\alpha) \log \frac{\beta}{1-\alpha} + \alpha \log \frac{1-\beta}{\alpha}}{p_0 \log \frac{p_1}{p_0} + (1-p_0) \log \frac{1-p_1}{1-p_0}}$$

2.2 $$E_{H_1}(n) = \frac{\beta \log \frac{\beta}{1-\alpha} + (1-\beta) \log \frac{1-\beta}{\alpha}}{p_1 \log \frac{p_1}{p_0} + (1-p_1) \log \frac{1-p_1}{1-p_0}}$$

It is not surprising 2.1 and 2.2 are near the expected sample size for the Sequential Sample Spacings test since the random variable d_n, defined in chapter I, is approximately binomial.

An example will make the stated property more clear.

Let F and G be N(0,1). $E_{H_0^*}(n)$ is the expected sample size of the Sequential Sample Spacings test. The alternative H_1^* is that F_1 is N(0,1), G_1 is N(1,1). $p_1 = p_1(F_1, G_1)$ (see chapter I) when F_1 is N(0,1) and G_1 is N(1,1) equals .5839. $p_0 = .5000$. Let $\alpha = \beta$. Using now, $p_1 = .5839$ and $p_0 = .5000$, $E_{H_0}(n)$ for Wald's test is given by 2.1 and is equal to

$$H(\alpha) = \frac{(1-\alpha)\log\frac{\alpha}{1-\alpha} + \alpha \log\frac{1-\alpha}{\alpha}}{(-\ .018)}$$

For $\epsilon_0 > 0$ then, there exists $\delta_0 > 0$ such that

$$\left| \frac{E_{H_0^*}(n)}{H(\alpha)} - 1 \right| < \epsilon_0$$

whenever $\alpha < \delta_0$.

The present study is to determine whether $\dfrac{E_{H_i^*}(n)}{E_{H_i}(n)}$ is near one for the usually encountered α values. The following table presents simulation results which verify the property when sampling from F and G each N(0,1) and testing H_0: F = G (vs) H_1: F = F_1:N(0,1), G = G_1:N(1,1).

TABLE I

	I	II	III	IV
	Average achieved sample size for the S.S.S. test when sampling from F and G each N(0,1) and testing H_0^*: F=G (vs) H_1^*: $F=F_1$: N(0,1), $G=G$: N(1,1) where $p_1=p_1(F_1,G_1) = .5839$	# of runs	Expected sample size for Wald's test using 2.1 with $p_0=.5000$ and $p_1=.5839$	$\frac{E_{H_0^*}(n)}{H(\alpha)}$, i.e. column I divided by column III.
$\alpha=.05$	137.5	103	185.57	.740
$\alpha=.01$	236.7	102	315.34	.750
$\alpha=.001$	370.7	101	482.68	.760
$\alpha=.0001$	503.0	89	644.83	.780
$\alpha=.00001$	632.2	81	806.61	.783

Using the average achieved sample size as an estimate of $E_{H_0^*}(n)$, the ratios in column IV, although less than .8, suggest the proper convergence from below. The rate of convergence, however, at least for the tested range of α values, is rather slow.

This observed convergence to one from below rather than above suggests 2.1 as an upper bound to $E_{H_0^*}(n)$. (It is important to point out that properties true under H_0:F = G for the Sequential Sample Spacings test are true universally, i.e., regardless of the form of F and G. Therefore, even though, under H_0 F and G were N(0,1) for the simulation, 2.1 may be used as an upper bound to $E_{H_0^*}(n)$ when sampling from any two identical distributions and testing an alternative which gives $p_1 = .5839$.)

For any sampling configuration such that $p_1 = p_1(F_1,G_1) = .5839$ column III of the table gives an upper bound to $E_{H_0^*}$. For example, with $\alpha = .05$ column III gives 185.57 as an upper bound to $E_{H_0^*}$ when testing $H_0:F = G$ (vs) H_1: $F = F_1:N(0,1)$, $G = G_1:N(1,1)$. The average achieved sample size given in column I when F and G are N(0,1) is 137.5. The table shows similar bounds for α = .01, .001, .0001, .00001.

The next table presents the same kind of simulation results when sampling from F:N(0,1), G:N(1,1) and testing $H_0:F = G$ (vs) $H_1:F = F_1:N(0,1)$, $G = G_1:N(1,1)$.

TABLE II

	I	II	III	IV
	Average achieved sample size for S.S.S. test when sampling from F:N(0,1), G:N(1,1) and testing $H_0^*:F = G$ (vs) $H_1^*:F=F_1$: N(0,1), $G=G_1:N(1,1)$. Where $p_1 = p_1(F_1,G_1) = .5839$.	# of runs	Expected sample size for Wald's test using 2.2 with $p_0 = .5000$ and $p_1 = .5839$	$\frac{E_{H_1}*(n)}{E_{H_1}(n)}$, i.e., column I divided by column III
α=.05	78	28	187.34	.41
α=.01	114	27	318.36	.35
α=.001	161	27	487.30	.33
α=.0001	213	26	650.99	.32
α=.00001	261	26	813.89	.32

There is no indication of convergence for this range of α values. The implication is that the convergence result is useful only for extremely small values of α. As before 2.2 may be used as an upperbound to $E_{H_1}*(n)$, but only for the above sampling configuration; that is to say, only when sampling and testing as described in column I. (2.2 may not be an upper bound to $E_{H_1}*(n)$ if sampling is from two differing distributions other than

$F:N(0,1)$, $G:N(1,1)$, i.e., properties under H_1 may not be universal even for distributions giving $p_1 = .5839$.)

Notice that the achieved sample size of the Sequential Sample Spacings test as shown in column I of table II is small relative to that given by 2.2 and shown in column III. This is related to the observed behavior of a random variable P_n. P_n will be defined and its behavior analyzed in the section entitled "A Convergence Result".

Discussion of Error Probability

Of interest to the study is how the achieved error probability of the Sequential Sample Spacings test varies as a function of the desired error probability. The achieved error probability has been proved to approach zero as α approaches zero. The relative rates of convergence however, are not known theoretically. From the simulation no explicit functional relationship between the achieved and the desired error probabilities was indicated. The results did suggest however that the achieved error probability is less than that which is sought.

Results for two sampling configurations are shown in the next tables.

TABLE III

Sampling from $F:N(0,1)$ $G:N(0,1)$ and testing $H_0:F=G$ (vs) $H_1:F=F_1$: $N(0,1)$ $G=G_1:N(0,1)$	α	# of runs	# of incorrect decisions	Pt. estimate of achieved error prob.	Upper 95% confidence bound on achieved error probability
	.05	103	0	0	.03
	.01	102	0	0	*
	.001	101	0	0	*
	.0001	89	0	0	*
	.00001	81	0	0	*

TABLE IV

Sampling from F:N(0,1) G:N(1,1) and testing H_0:F=G (vs) H_1:F=F_1: N(0,1), G= G_1:N(0,1)	α	# of runs	# of incorrect decisions	Pt. estimate of achieved error prob.	Upper 95% confidence bound on achieved error probability
	.05	28	0	0	*
	.01	27	0	0	*
	.001	27	0	0	*
	.0001	26	0	0	*
	.00001	26	0	0	*

The upper 95% confidence bound was determined by solving $(1-\alpha_A)^N = .05$ where α_A is the error probability achieved and N is the number of simulation runs made for the given α.

*Not enough simulation runs were made to give an upper bound to α_A less than the desired α.

Other Alternative Hypotheses

One simulation run was made when sampling from F:N(0,1), G:N(1/2,1) and testing H_0:F = G (vs) H_1:F = F_1:N(0,1), G = G_1:N(1/2,1); when sampling from F:N(0,1), G:N(1/4,1) and testing H_0:F = G (vs) H_1:F = F_1:N(0,1), G = G_1:N(1/4,1); and when sampling from F:N(0,1), G:N(1/8,1) and testing H_0:F = G (vs) H_1:F = F_1:N(0,1), G = G_1:N(1/8,1).

These sampling configurations were used primarily to investigate P_n for differing alternatives.

The values of p_1 are .5142, .5037, .5010 respectively. The α's used were again .05, .01, .001, .0001, .00001. The test, however, could terminate in the computer time available for each run only for α = .05 and $p_1 \doteq .5142$. The decision made for this case was correct.

A Convergence Result

Consider the random variable P_n defined in the following manner.

$$P_n = \Sigma\, G(Y_{n,i}) - G(Y_{n,i-1}), \text{ with summation over}$$

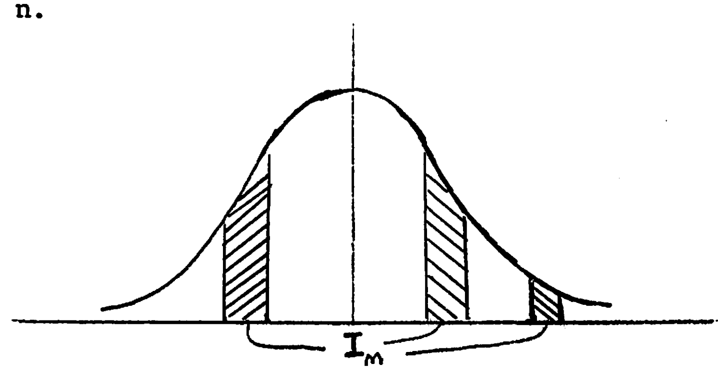
all intervals $(Y_{n,i-1}, Y_{n,i}] \in I_n$.

P_n is the probability that $Y_{n+1} \in I_n$, i.e., P_n is the probability of "success" at stage n where a "success" causes a vertical increment of one unit for the random variable d_n, at stage n. (See chapter I).

Example Showing Construction of P_n.

Let G be N(0,1). Suppose at sampling stage n, the critical region I_n is that shown in figure 2. The shaded area then determines P_n for stage n.

2.



Blumenthal has shown that P_n converges in probability to p_1, $(P_n \xrightarrow{P} p_1)$, where $p_1 = p_1(F,G)$. F and G are the parent populations from which samples are taken.* [See chapter I for construction of $p_1(F,G)$.]

For any F, G such that F = G, $p_1 = p_1(F,G) = \frac{1}{2} = p_0$.

The value of P_n was computed every 50th sampling stage of each simulation run.

*Keep in mind that F and G are known for the simulation.

These computations indicate that when sampling from F and G both N(0,1) P_n is near 1/2 for reasonably small sample sizes.

The analysis of P_n when sampling from F and G both N(0,1) is tabulated below.

TABLE V

Sampling stage n	# of observations k, of P_n taken	Est. of $Var_{H_0}(P_n)$	Est. of standard deviation of P_n under H_0	Est. of $E_{H_0}(P_n)$	Lower bd. of 99% conf. interval for $E_{H_0}(P_n)$	Upper level of 99% conf. interval for $E_{H_0}(P_n)$
50	104	.00131	.03620	.48944	.47860	.50026
100	104	.00059	.02428	.49493	.48764	.50222
150	104	.00038	.01949	.49594	.49009	.50179
200	104	.00029	.01703	.49871	.49370	.50372
250	103	.00028	.01676	.49837	.49342	.50333
300	100	.00025	.01583	.49845	.49370	.50320
350	97	.000193	.01390	.49944	.49521	.50368
400	89	.000196	.01401	.49994	.49548	.50439
450	86	.000199	.01411	.50008	.49551	.50464
500	72	.000156	.01251	.49849	.49406	.50291
550	62	.000141	.01188	.49943	.49490	.50396
600	54	.000119	.01094	.49976	.49529	.50423
650	47	.000115	.01074	.50013	.49543	.50483
700	38	.000114	.01070	.49953	.49432	.50474
750	31	.000051	.00719	.49982	.49452	.50512
800	16	.000027	.00525	.50056	.49669	.50443
850	16	.000040	.00637	.50181	.49712	.50650
900	16	.000031	.00557	.50199	.49789	.50609
950	13	.000029	.00541	.50099	.49640	.50558
1000	10	.000021	.00468	.49894	.49414	.50371
1050	7	.000041	.00645	.49755	.48853	.50657
1100	5	.000037	.00615	.49735	.48475	.50995
1150	3	.000046	.00679	.49703	.45903	.53503

$\overline{P}_n = \frac{1}{k}\sum_{i=1}^{k} P_{n,i}$ estimates $E(P_n)$, where $P_{n,1}, P_{n,2}, \ldots, P_{n,k}$ are k independent observations on P_n. $s^2_{P_n} = \frac{1}{k-1}\sum_{i=1}^{k}(P_{n,i} - \overline{P}_n)^2$ is the estimate of $Var(P_n)$ used. $s_{P_n} = \sqrt{s^2_{P_n}}$ is the estimate of standard deviation of P_n used. $(n = 50, 100, \ldots, 1150.)$

Confidence intervals for $E(P_n)$ were determined in the following manners:

For $31 \leqq k \leqq 104$, and $n = 50, 100, \ldots, 1150$, $\frac{\overline{P}_n - E(P_n)}{\sigma_{P_n}/\sqrt{k}}$ is approximately $N(0,1)$ by the Central Limit Theorem. Since $s_{P_n} \xrightarrow{P} \sigma_{P_n}$,

$\frac{\overline{P}_n - E(P_n)}{s_{P_n}/\sqrt{k}} \xrightarrow{P} \frac{\overline{P}_n - E(P_n)}{\sigma_{P_n}/\sqrt{k}}$, and $\frac{\overline{P}_n - E(P_n)}{s_{P_n}/\sqrt{k}}$ is distributed approximately $N(0,1)$. Probability statements may be made:

$$P\{-3 < \frac{P_n - E(P_n)}{s_{P_n}/\sqrt{k}} < 3\} \cong .99 \qquad P\{\overline{P}_n - 3s_{P_n}/\sqrt{k} < E(P_n) < 3s_{P_n}/\sqrt{k} + \overline{P}_n\} = .99$$

For $3 \leqq k \leqq 31$, n exceeds 750. By the construction of P_n, it is the sum of n individual probabilities. By a theorem proved in [3] P_n is approximately normal for large n. Therefore

$\frac{P_n - E(P_n)}{\sigma_{P_n}/\sqrt{k}}$ is approximately $N(0,1)$. Since P_n is approximately normal,

$\frac{(k-1)s^2_{P_n}}{\sigma^2_{P_n}}$ is approximately $\chi^2_{(k-1)}$. Therefore $\frac{\overline{P}_n - E(P_n)}{s_{P_n}/\sqrt{k}}$

is approximately distributed $t_{(k-1)}$. The following probability statement may be made: $P\{-t_{(k-1)} < \frac{P_n - E(P_n)}{s_{P_n}/\sqrt{k}} < t_{(k-1)}\} \cong .99$. Values of $t_{(k-1)}$ may be obtained giving confidence intervals for $E(P_n)$.

Table V shows the estimate of $Var_{H_0}(P_n)$ is tending toward zero as n gets large. By Chebishev's inequality, $P\{|P_n - E_{H_0}(P_n)| > \epsilon\} < \frac{Var_{H_0}(P_n)}{\epsilon^2}$. This implies, since the estimated variance of P_n tends to zero, that $P_n \xrightarrow{p} E_{H_0}(P_n)$.

The simulation indicates P_n is near $E_{H_0}(P_n)$ for reasonably small n, since the estimate of $Var_{H_0}(P_n)$ is near zero for small n. For example, with n = 600 the estimated variance of P_n is .000119.

We know $P_n \xrightarrow{p} 1/2$. The simulation results indicate $P_n \xrightarrow{p} E_{H_0}(P_n)$. This says, eventually $E_{H_0}(P_n) \to 1/2$. The rate at which $E_{H_0}(P_n)$ converges to 1/2 must be fairly rapid, since all computed confidence intervals for $E_{H_0}(P_n)$ include the value 1/2. The probability of all computed confidence intervals covering 1/2 is small if $E_{H_0}(P_n)$ is bounded appreciably away from 1/2. Notice also, the estimates of $E_{H_0}(P_n)$ are very near .5000 for the range of n studied.

When sampling from F:N(0,1), G:N(1,1), p_1 = .5839. The following table gives information related to P_n in this case.

TABLE VI

I Sampling stage n	II # of obs. k, taken of P_n	III Est. of $Var_{H_1}(P_n)$	IV Est. of standard deviation of P_n under H_1	V Est. of $E_{H_1}(P_n)$	VI Lower bound of 99% conf. interval for $E_{H_1}(P_n)$	VII Upper bound of 99% conf. interval for $E_{H_1}(P_n)$
50	17	.00185	.04308	.70101	.67063	.73139
100	17	.00041	.02020	.70082	.68658	.71506
150	17	.00050	.02240	.70647	.69077	.72217
200	16	.00022	.01510	.70406	.69168	.71644
250	7	.00023	.01520	.70000	.67878	.72122
300	4	.00011	.01070	.70483	.67380	.73586
350	3	.00046	.02160	.70216	.61376	.79056

$P_n \xrightarrow{p} .5839$. However, for the range of n observed, the average value of P_n is near .70 as shown in column V. In fact, a linear least squares line giving $\overline{P}_n$ as a function of n, and using the numbers in columns I and V, has a positive slope equal .00077. This implies P_n must converge to .5839 only for extremely large n. Notice, none of the confidence intervals for $E_{H_1}(P_n)$ do in this case cover .5839. This is very strong indication $E_{H_1}(P_n)$ is appreciably different from .5839 for the range of n studied.

The high observed value of P_n and the relatively small sample sizes at termination when sampling from F:N(0,1), G:N(1,1) are related. Recall P_n is the probability of "success" at stage n, or the probability that d_n takes a vertical increment at stage n. If this probability is high d_n grows at a rapid rate and the test is terminated early.

For each of the next configurations only one run was made.

When sampling from F:N(0,1) and G:N(1/2, 1) $P_n \xrightarrow{p} .5142$. In this case, the observed values of P_n for $n \leq 600$ are near .56 which is close to the limit .5142.

For F:N(0,1) and G:N(1/4,1) P_n is observed to be near the appropriate p_1 = .5037 for n ≦ 800. The same is true for F:N(0,1) and G:N(1/8,1) where the appropriate p_1 is .5010.

The above results suggest for F and G normal the rate of convergence of P_n to p_1 is a decreasing function of the difference in location parameters of F and G. It is also possible the rate of convergence for P_n depends on the shape of F and G.

Table VII gives, for the last three sampling configurations mentioned, an indication of the behavior of P_n.

TABLE VII

n	P_n for F:N(0,1) G:N(1/2,1)	P_n for F:N(0,1) G:N(1/4,1)	P_n for F:N(0,1) G:N(1/8,1)
50	.62703	.50663	.51633
100	.55784	.46863	.55787
150	.59788	.47183	.53463
200	.61154	.53564	.51218
250	.57261	.51550	.48798
300	.55673	.50989	.49704
350	.55422	.51422	.49767
400	.57022	.51058	.49411
450	.56786	.51493	.49472
500	.56279	.51821	.50205
550	.56452	.51344	.51603
600	.55454	.51517	.50928
650	.56040	.52074	.50668
700	.56351	.52876	.50599
750	.56791	.53086	.50368
800	.57442	.53151	.49715
900	.56418	.52450	.50111
950	.56709	.52223	.50121
1000	.56704	.52044	.50115

Comparing with Lehmann's Test

It is of interest to compare the Sequential Sample Spacings test with a similar test by Lehmann [2].

Lehmann's test also relates to a two sample problem and tests $H_0:F = G$ (vs) $H_1:F \neq G$. It considers the number of quadruples $X_{2-1}, X_{2n}; Y_{2n-1}, Y_{2n}$ for which either the two X's fall below the two Y's or vice-versa, $n = 1,2,\ldots$. The probability of this event for any n is given by

$$p = \frac{1}{3} + 2\int_{-\infty}^{\infty} (F-G)^2 \, d\left(\frac{F+G}{2}\right),$$

and may be considered the probability of success on independent trials. Therefore; $H_0:F = G$ (vs) $H_1:F \neq G$ reduces to testing $H_0:p = p_0 = \frac{1}{3}$ (vs) $H_1:p = p_1 > \frac{1}{3}$.

A sequential formulation follows for testing the above $H_0:p = p = \frac{1}{3}$ (vs) $H_1:p = p_1 > \frac{1}{3}$. Let d_n^* be the number of successes in n independent trials where a success has probability p. Observing $X_{2n-1}, X_{2n}; Y_{2n-1}, Y_{2n}$ constitutes the n'th trial. Let a_n^* and r_n^* be given by 1.1 and 1.2 respectively. (In formulas 1.1 and 1.2 $p_0 = \frac{1}{3}$. p_1 is determined by the specific alternative distributions F_1 and G_1 in mind and is given by

$$p_1 = \frac{1}{3} + 2\int_{-\infty}^{\infty} (F_1 - G_1)^2 \, d\left(\frac{F_1 + G_1}{2}\right) dx \text{ .)}$$

The test is accept H_0 if $d_n^* \leq a_n^*$, reject H_0 if $d_n^* \geq r_n^*$ and continue sampling if $a_n^* < d_n^* < r_n^*$.

In comparing the tests, 2.1 and 2.2 may be used to compute the approximate expected sample size for the Sequential Sample Spacings test and the exact expected sample size for the test by Lehmann.*

*2.1 and 2.2 are approximate for the S.S.S. test since d_n is only approximately binomial; for Lehmann's test d_n^* is exactly binomial.

Recall $p_1 = p_1(F_1, G_1) = .5839$ for the Sequential Sample Spacings test when under H_1 we have $F = F_1{:}N(0,1)$, $G = G_1{:}N(1,1)$. $p_0 = .5000$.

$$p_1 = \frac{1}{3} + 2\int_{-\infty}^{\infty} (F_1 - G_1)^2 \, d\left(\frac{F_1 + G_1}{2}\right) dx = .4938 \text{ for Lehmann's test.}$$

$p_0 = \frac{1}{3}$ for the same sampling configuration. Some comparative results of the simulation are tabulated below.

TABLE VIII

	I Approximate expected sample size for the S.S.S. test by 2.1 when sampling from F=G using p_1=.5839	II Average achieved sample size of S.S.S. test when sampling from F and G:N(0,1) and testing H_0:F=G (vs) H_1:F=F_1: N(0,1), G = G_1:N(1,1)	III Exact expected sample size of Lehmann's test using 2.1 when sampling from F and G:N(0,1) and using p_1=.4931	IV Ratio of numbers in columns II and III.	V Ratio of numbers in columns I and III.
α=.05	185.7	137.5	50.39	2.72	3.68
α=.01	315.34	236.7	85.64	2.76	3.68
α=.001	482.68	370.7	131.09	2.82	3.68
α=.0001	644.83	503.0	175.12	2.87	3.68
α=.00001	806.61	632.0	218.94	2.88	3.68

TABLE IX

	I Approximate expected sample size for the S.S.S. test by 2.2 when sampling from F: N(0,1), G:N(1,1) and testing H_0: F=G (vs) H_1:F=F_1: N(0,1),G=G_1:N(1,1)	II Average achieved sample size for S.S.S. test when sampling from F: N(0,1) and G:N(1,1) and testing H_0:F=G (vs) H_1:F=F_1:N(0,1) G=G_1:N(1,1)	III Exact expected size for Lehmann's test when F=F_1: N(0,1) and G=G_1: N(1,1) using 2.2 and p_1=.4938	IV Ratio of numbers in columns II and III.	V Ratio of numbers in columns I and II.
α=.05	187.34	78	48.96	1.62	3.82
α=.01	318.36	114	82.35	1.38	3.86
α=.001	487.30	161	126.07	1.27	3.86
α=.0001	650.99	213	168.07	1.26	3.85
α=.00001	813.89	261	210.56	1.23	3.86

When sampling from $F = G$ and testing $H_0: F = G$ (vs) $H_1: F = F_1: N(0,1)$, $G = G_1: N(1,1)$, the sequence formed by the ratio of the achieved sample size of the Sequential Sample Spacings test to that predicted for Lehmann's by 2.1 is tending toward 3 as shown in column IV, Table VIII. The ratio of the approximate expected sample size of the Sequential Sample Spacings test, given by 2.1, to that given by 2.1 for Lehmann's test is near 4 as shown in column V, Table VIII. In other words, the ratio of the sample size when sampling from $F = G$ and testing $H_0: F = G$ (vs) $H_1: F = F_1: N(0,1)$, $G = G_1: N(1,1)$ of Lehmann's test relative to the Sequential Sample Spacings test is shown to be near $\frac{1}{3}$ by the simulation and near $\frac{1}{4}$ by formula 2.1.

This result does not imply the expected sample size of Lehmann's test under H_0, is less than that of the Sequential Sample Spacings test for all distributions, since the expected sample size for each test depends on the specific alternative H_1 that is chosen. For H_1's of the form $F = F_1: N(0,1)$, $G = G_1: N(\delta,1)$ Lehmann's test is likely to have uniformly smaller expected sample size.

The simulation results indicate that Lehmann's test has a smaller sample size at termination than the Sequential Sample Spacings test; both when sampling from $F = G$ and from $F = F_1: N(0,1)$, $G = G_1: N(1,1)$ for the specific alternative chosen.

It is not really fair, though, to compare the average sample size of the Sequential Sample Spacings test with that predicted for Lehmann's by 2.1 and 2.2, without considering the achieved error probabilities of each test. Lehmann's test achieves an error probability equal to the α built into its bounds, i.e., a_n^* and r_n^*, whereas the Sequential Sample Spacings test achieves an error probability less than the built in α, at least when sampling from $F = G$.

Recall when sampling from $F = G$ and testing $H_0{:}F = G$ (vs) $H_1{:}F = F_1{:}N(0,1)$, $G = G_1{:}N(1,1)$ the Sequential Sample Spacings test achieved an error probability less than or equal .03 with $\alpha = .05$. This desirable property suggests that when comparing it with the test by Lehmann, an upper bound to the error probability actually achieved by the Sequential Sample Spacings test should be used in 2.1 and 2.2 to determine Lehmann's expected sample size. For example, if $\alpha = .05$ is used for the Sequential Sample Spacings test of $H_0{:}F = G$ (vs) $H_1{:}F = F_1{:}N(0,1)$, $G = G_1{:}N(1,1)$, an $\alpha = .03$ must be used in 2.1 and 2.2 to predict Lehmann's expected sample size.

Even after adjusting the error probabilities Lehmann's test has smaller expected sample size when sampling from $F = G$, and testing $H_0{:}F = G$ (vs) $H_1{:}F = F_1{:}N(0,1)$, $G = G_1{:}N(1,1)$. For example, with $\alpha = .05$ the achieved sample size of the Sequential Sample Spacings test is 137.5. Using $\alpha = .03$, 2.1 predicts Lehmann's test to have an expected sample size less than 85.

When sampling from $F{:}N(0,1)$, $G{:}N(1,1)$ and testing $H_0{:}F = G$ (vs) $H_1{:}F = F_1{:}N(0,1)$, $G = G_1{:}N(1,1)$ the Sequential Sample Spacings test compares quite favorably with Lehmann's. For $\alpha = .05$, and assuming an achieved error probability equal .03, the Sequential Sample Spacings test has an achieved average sample size equal to 78. Lehmann's test with $\alpha = .03$ has an expected sample size around 60.

CHAPTER III

The Simulator

This section is devoted to a discussion of the simulator used in carrying out the experiment. A print out of the simulator is included.

The simulator is written in Fortran and Fortran Symbolic programming language for use on a Control Data 1604 computer. It uses the Sequential Sample Spacings test to test H_0: $F=G$ (vs) H_1: $F=F_1$, $G=G_1$ for five distinct values of the confidence level α. The simulator is built with $\alpha = \beta$.

Any alternative hypothesis H_1, may be tested by determining the appropriate p_1 and setting the parameter PPP of the program equal to p_1.

Built into the simulator is a mechanism for generating $N(0,1)$ random variables. This generator is known as RANDEV and is really a library subroutine available on magnetic tape.*

For the simulation study RANDEV generated both F and G. However, the populations F and G from which samples are taken may have any desired form by replacing RANDEV with the appropriate generators.

The generator RANDEV depends upon another subroutine RAND1. RAND1 generates uniform $[0,1]$ variates. Each variate from RANDEV is formed by generating 16 uniform $[0,1]$ variates from RAND1. RANDEV is set equal to

$$\frac{\sum_{i=1}^{16} x_i - 8}{\sqrt{16/12}}$$

which is approximately $N(0,1)$

*A subroutine is a program whose services may be called upon by making a Fortran Statement. In this case the statement is RANDEV(-1).

by the Central Limit Theorem.

RAND1 generates a pseudo sequence of random numbers uniformly distributed on $[0,1]$ by using a modulo technique. These numbers have many of the properties of randomness [4].

It is likely desired forms of F and G other than normal may be obtained by transforming variates from RAND1.

Recall that the Sequential Sample Spacings test requires the ordering of the observed X's and Y's and the $S_{n,i}$. (see chapter I) Two ordering techniques are built into the simulator. The first of these applies a subroutine called NSORT. NSORT orders N numbers in either increasing or decreasing order by a frequency technique. At the same time an equivalent permutation of the numbers $1,2,\ldots,N$ is performed. For example, if NSORT were called upon to put in increasing order the numbers $2_1, 4_2, 1_3, 7_4, 3_5, 6_6, 5_7$, they would be ordered $1,2,3,\ldots,7$ and the integer subscripts would be arranged as follows: 3,1,5,2,7,6,4. This second sequence of numbers tells us that 1 was in the 3rd position of the original unordered sequence, 2 was in the first position,etc.

The permutation of the integers $1,2,\ldots,N$ performed by NSORT is invaluable to simulating the Sequential Sample Spacings test. Consider at stage n=4, $S_1(4)$, $S_2(4)$, $S_3(4)$, $S_4(4)$, $S_5(4)$ ordered to give $S_5(4)=S_{4,1} \leq S_2(4)=S_{4,2} \leq S_1(4)=S_{4,3} \leq S_4(4)=S_{4,4} \leq S_3(4)=S_{4,5}$. The integers $1,2,\ldots,5$ would be permuted to 5,2,1,4,3.

Suppose $\frac{1}{17}\sum_{i=1}^{2}(S_{4,i}+1) \leq \frac{1}{2} < \frac{1}{17}\sum_{i=1}^{3}(S_{4,i}+1)$.

Then $J_4=2$, which means I_4 is the union of the two intervals corresponding to $S_5(4)$ and $S_2(4)$. Of course, in the computer we cannot simply look to see which symbols correspond when determining I_4. We must check the first $J_4=2$ elements of 5,2,1,4,3. This specifies completely I_4.

The routine NSORT is very efficient when the numbers to be ordered have no previous order, i.e., they are completely disarranged. However, NSORT does not take advantage of any order already present in the numbers. This makes NSORT inefficient as a mechanism for ordering the X's and Y's at each sampling stage n, for ordering of the X's at stage n simply requires that three new observations be inserted with the already ordered 3(n-1) X's. For the case of Y at stage n, only one new observation must be put in its appropriate position.

An ordering technique which utilizes a machine command called Thresh-Hold-Search, THS(z), is most efficient for ordering the X's and Y's at each sampling stage n. THS(z) searches a list of numbers until it finds one larger than z. z may then be inserted in its proper position. Three applications of THS(z) puts X_{3n-2}, X_{3n-1}, X_{3n} in their proper position relative to the 3(n-1) X's already in order. One application of THS(z) at each stage n puts the Y's in proper order.

The simulator was originally built with NSORT as the exclusive ordering mechanism. With THS(z) used to order the X's and Y's, the expected running time of a simulation was reduced nearly 60%.

In the evaluation of P_n when G is $N(0,1)$, (see chapter II), a subroutine denoted CDFN(z) is used.

$$\text{CDFN}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt .$$

CDFN(z) is accurate to the tenth decimal. This degree of accuracy is more than adequate since P_n need be accurate to no more than five decimals.

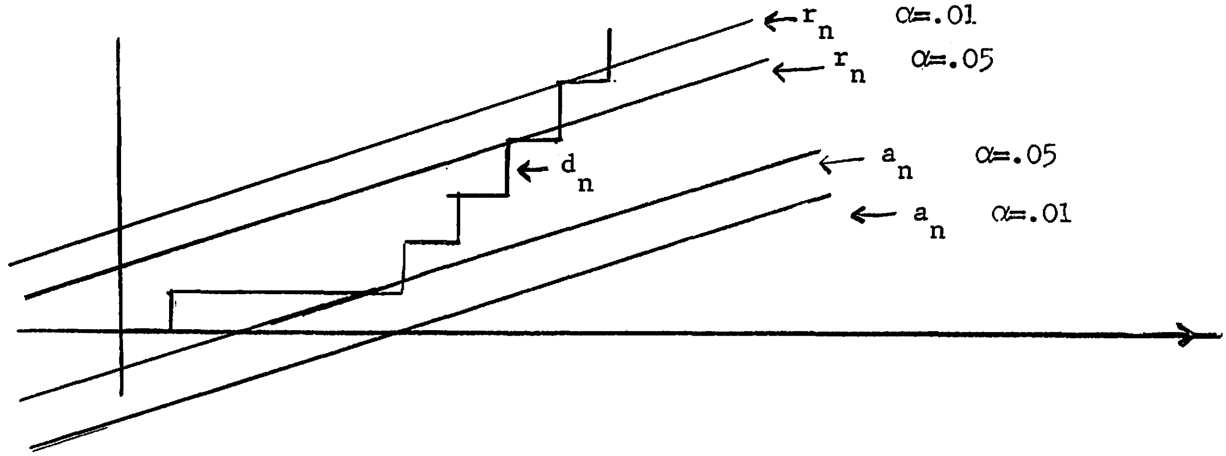
The behavior of P_n for distributions other than normal may be of interest to a researcher. For example, the rate of convergence of P_n to the appropriate p_1 when sampling from differing exponential distributions may be compared with the convergence rate for identical exponentials.

P_n for the exponential case may be studied by replacing CDFN(z) by a subroutine which computes the cumulative distribution function for an exponential variate.

If the cumulative distribution function to be computed when studying P_n for distributions other than normal, involves an integral with no analytic solution, numerical solution techniques are available which are fast and accurate.

The Numerical Analysis Center at the University of Minnesota has available a routine for numerical integration of any reasonably smooth curve which is accurate to nine decimals.

Earlier it was stated the simulator tests H_0: $F=G$ (vs) H_1: $F=F_1$, $G=G_1$ for five distinct values of α. It is worth pointing out that a decision to accept H_0 for $\alpha=.05$ may be followed by a rejection of H_0 for $\alpha=.01$ or vice-versa. The next figure shows the situation.



An error could be made by assuming acceptance of H_0 for $\alpha=.05$ implies acceptance for $\alpha=.01$. The simulator is designed to test H_0: F=G (vs) H_1: $F \neq G$ at each sampling stage for all α's for which a final decision has not been made.

The simulator, as it is presented in the next few pages, will operate only on Control Data computers. However, the instructions written in machine coding may be replaced by equivalent instructions for other machines or by Fortran statements, to make the simulator compatible with most Fortran systems.

Simulator Program

```
      DIMENSIONX(6000),  Y(2001),  S(2001), IC(2001), Z(2001), P(200),
     1  NTEMP(257)  ,  X1(6000) ,  R(3),  RI(3) ,  R2(1)
      COMMON X, Y, IC, Z, P, NTEMP, X1, R, RI, R2  ,S
      A=.05
      IIJJ=12
      PPP=.5037
      HACK=.0
      AA=.01
      AAA=.001
      AAB=.0001
      AAC=.00001
      DDA2= LOGF(PPP/(1.-PPP))
      DD1=(LOGF(A/(1.-A)))/DDA2
      DDAA1=(LOGF(AA/(1.-AA)))/DDA2
      DDAAA1= (LOGF(AAA/(1.-AAA)))/DDA2
      DDAAB=(LOGF(AAB/(1.-AAB)))/DDA2
      DDAAC=(LOGF(AAC/(1.-AAC)))/DDA2
      DDD1=(LOGF(1./(2.*(1.-PPP))))/DDA2
      EEE1=(LOGF(1./(2.*(1.-PPP))))/DDA2
      EE1=(LOGF((1.-A  )/A  ))/DDA2
      EEAA1=(LOGF((1.-AA)/AA))/DDA2
      EEAAA1=(LOGF((1.-AAA)/AAA))/DDA2
      EEAAB=(LOGF((1.-AAB)/AAB))/DDA2
      EEAAC=(LOGF((1.-AAC)/AAC))/DDA2
```

```
      DD=0.0
      Q=RANDEV(IIJJ)
      A1=0.
      A2=0.
      A3=0.
      A4=0.
      A5=0.
      Y(1)= RANDEV(-1)
      II=0
      LL=1
      LLL=LL+2
      M=1
      PRINT  91,  EE1, EEAA1, EEAAA1, EEAAB, EEAAC
      DO 10  N=1, 2000
      B=N
      NN1=N-1
      NN=3*N
      B4=4.*B+1 .
      N2=N+1
      IF(N-1)  73, 73, 74
   73 DO 900 J=LL, LLL
  900 X(J)=RANDEV(-1)
      LL=LL+3
      LLL=LL+2
   80 FORMAT(10E12.4, /)
      CALL NSORT(Y, Z, IC, N, 0, +6, NTEMP)
```

```
      CALL NSORT(X, X1, IC, NN, 0, +6, NTEMP)
      GO TO 3
74    DO 75 K=1,  3
 75   R(K)=RANDEV(-1)
      CALL NSORT(R, RI, IC, 3, 0, -6, NTEMP)
      NXZ=0
      MN3=NN-3
      MN2=NN-2
      MN1=NN-1
      LDA(R+1)    LIL2(MN3).
      THS2(X+1)    SLJ0(84).
      SIL2(NXZ)     SLJ0(85).
 95   LDA(R+1)   STA2(X+2).
      LDA(R+2)    INI2(1).
      THS2(X+1)   SLJ0(86).
      SIL2(NXZ)    SLJ0(96)     .
 89   LDA(R+2)    STA2(X+2).
      LDA(R+3)    INI2(1)   .
      THS2(X+1)    SLJ0(99)    .
      SIL2(NXZ)    SLJ0(93) .
 98   LDA(R+3)    STA2(X+2).
      GO TO 82
 93   NNL=MN1-NXZ-1
      DO 94 I=1,  NNL
      KI=MN1-I+1
94    X(KI+1)= X(KI)
```

```
      GO TO 98
   96 NNL=MN2-NXZ-1
      DO 88 I=1,  NNL
      KI=MN2-I+1
   88 X(KI+1)= X(KI)
      GO TO 89
   85 IF(NXZ-MN3+1)  508, 95, 508
  508 NNL=MN3-NXZ-1
      DO 83 I=1, NNL
      KI=MN3-I+1
  83  X(KI+1)=X(KI)
      GO TO 95
  84  DO 81  K=1, MN3
      KJ=MN3-K+1
   81 X(KJ+3)= X(KJ)
      X(3)=R(1)
      X(2)=R(2)
      X(1)=R(3)
      GO TO 82
  86  DO 87 K=1, MN2
      KJ=MN2-K+1
   87 X(KJ+2)= X(KJ)
      X(2)=R(2)
      X(1)=R(3)
      GO TO 82
   99 DO 92 K=1, MN1
```

```
      KJ=MN1-K+1
   92 X(KJ+1)= X(KJ)
      X(1)=R(3)
   82  CONTINUE
      LDA(R2+1)  LIL2(NN1).
      THS2(Y+1)   SLJ0(500).
      SIL2(NXZ)   SLJ0(502).
  504 LDA(R2+1)   STA2(Y+2).
      GO TO 505
  500 DO 501  K=1, NN1
      KJ=NN1-K+1
  501 Y(KJ+1)=Y(KJ)
      Y(1)=R2(1)
      GO TO 505
  502 IF(NXZ-NN1+1)  509,504,509
  509 NNL=NN1-NXZ-1
      DO 503 I=1, NNL
      KI=NN1-I+1
  503 Y(KI+1)=Y(KI)
      GO TO 504
  505  CONTINUE
    3 I1=1
      DO 20 J=1,N
      JK=N-J+1
      SKOUNT=0.
      DO  18  I=I1,  NN
```

```
      IK=NN-I+1
      IF(X(IK)-Y(JK))5, 5, 19
    5 SKOUNT=SKOUNT+1.
      IF (I-NN)  18, 24, 24
   18 CONTINUE
   19 S(J)=SKOUNT
      I1=I
   20 CONTINUE
      IR=3*N-I+1
      R=IR
      S(N+1)=R
      GO TO 26
   24 S(J)=SKOUNT
      KK= J+1
      N1= N+1
      JJ=J
      AJJ= JJ
      DO  25 K=KK,  N1
   25 S(K)=0.
   26 NI= N+1
  910 CALL NSORT(S, Z, IC, NI, -1, -6, NTEMP)
      TEST=0.
      DO 30 J=1, 2001
      TEST=TEST+(S(J)+1.)/(B4)
      IF(TEST-.5) 30, 30, 29
   30 CONTINUE
```

```
 29 JJ=J-1
    IF (N-1)  113,113,112
112 KKK=50*II
    IF (N-KKK) 110,111,110
113 DO 114 K=1, 200
114 P(K)=0
111 DO 100 K=1,  JJ
    L=IC(K)
    LML=N-L+1
    LM1=L-1
    LML1=LML+1
    IF(LM1     ) 101,102,101
102 P(M)=P(M)+CDFN(Y(N)-HACK)
    GO TO 100
101 IF( L    -N2)  103, 104, 103
104 P(M)=P(M)+1.-CDFN(Y(1)-HACK)
    GO TO 100
103  P(M)=P(M)+CDFN(Y(LML)-HACK)-CDFN(Y(LML1)-HACK)
100 CONTINUE
    PRINT  91, EVENT, B, P(M), E1, E101, E1001, E10001, E100001
    II=II+1
    M=M+1
110 CONTINUE
    BE=B*EEE1
    E1=EE1+BE
    E101=EEAA1+BE
```

```
      E1001=EEAAA1+BE
      E10001=EEAAB+BE
      E100001=EEAAC+BE
      D1=DD1+BE
      D101=DDAA1+BE
      D1001=DDAAA1+BE
      D10001=DDAAB+BE
      D100001=DDAAC+BE
      IF(N-1) 90,90,16
   90 EVENT=0.
   16 Y(N+1)=RANDEV(-1)
      R2(1)=Y(N+1)
      YN= Y(N+1)
      DO 40 I=1, JJ
      ICC=IC(I)
      ICK=N-ICC+1
       ICC1=ICC-1
      ICC1K=ICK+1
      IF(ICC-NI)  46, 47, 46
   47 IF(YN-Y(1))  40, 40, 57
   46 CONTINUE
      IF(ICC1)  56, 58, 56
   58 IF( YN-Y(N))  57, 57, 40
   56 CONTINUE
      IF(YN-Y(ICK))  55, 55, 40
   55 IF(YN-Y(ICC1K))  40, 40, 57
```

```
 40 CONTINUE
    GO TO 59
 57 EVENT=EVENT+1.
 59 CONTINUE
912 IF(A1-1.)  63, 17, 63
 63 IF(EVENT-D1)  60, 60, 62
 60 PRINT  70
 70 FORMAT( 1X,  36H ACCEPT HNOT FOR A=.05 DELTA=1.                    )
    A1=1.
    PRINT  80,   EVENT, B, D1, E1
    CALL TIME(1HP, 1H1 )
    GO TO 300
 62 IF(EVENT-E1)  300, 64, 64
 64 PRINT  72
 72 FORMAT(1X,  36H REJECT HNOT FOR A=.05 DELTA=1.                     )
    A1=1.
    PRINT  80, EVENT, B, D1, E1
    CALL TIME(1HP, 1H1 )
    GO TO 300
 17 CONTINUE
    IF(A2-1.)  163, 27, 163
163 IF(EVENT-D101)  160, 160, 162
160 PRINT  170
170 FORMAT(1X,  36H ACCEPT HNOT FOR A=.01 DELTA=1.                     )
    A2=1.
    PRINT  80, EVENT, B, D101, E101
```

```
      CALL TIME(1HP, 1H1 )
      GO TO 300
  162 IF(EVENT-E101)  300, 164, 164
  164 PRINT 172
  172 FORMAT( 1X,  36H REJECT HNOT FOR A=.01  DELTA=1.                  )
      A2=1.
      PRINT 80, EVENT, B, D101, E101
      CALL TIME(1HP, 1H1 )
      GO TO 300
   27 CONTINUE
      IF(A5-1.)  263, 200, 263
  263  IF(EVENT-D1001)  260, 260, 262
  260 PRINT 270
  270 FORMAT(1X,  36H ACCEPT HNOT FOR A=.001 DELTA=1.                    )
      PRINT  80,  EVENT, B, D1001, E1001
      CALL TIME(1HP, 1H1 )
      A5=1.
      GO TO 300
  262 IF(EVENT-E1001)  300, 264, 264
  264 PRINT 272
  272 FORMAT( 1X, 36H REJECT HNOT FOR A=.001 DELTA=1.                    )
      PRINT  80, EVENT, B, D1001, E1001
      CALL TIME(1HP, 1H1 )
       A5=1.
      GO TO 300
  200 CONTINUE
```

```
      IF(A3-1.)  363, 317, 363
  363 IF(EVENT-D10001)  360, 360, 362
  360 PRINT 370
  370 FORMAT(1X,  36H ACCEPT HNOT FOR A=.0001 DELTA=1.                    )
      A3=1.
      PRINT  80, EVENT, B, D10001, E10001
      CALL TIME(1HP, 1H1 )
      GO TO 300
  362 IF(EVENT-E10001)  300, 364, 364
  364 PRINT 372
  372 FORMAT(1X, 36H REJECT HNOT FOR A=.0001 DELTA=1.                     )
      A3=1.
      PRINT  80, EVENT, B, D10001, E10001
      CALL TIME(1HP, 1H1 )
      GO TO 300
  317 CONTINUE
  463 IF(EVENT-D100001)  460, 460, 462
  460 PRINT 470
  470 FORMAT( 1X,  36H ACCEPT HNOT FOR A=.00001 DELTA=1.                  )
      PRINT  80, EVENT, B, D100001, E100001
      CALL TIME(1HP, 1H1 )
      GO TO 11
  462 IF(EVENT-E100001)  300, 464, 464
  464  PRINT 472
  472 FORMAT(1X,  36H REJECT HNOT FOR A=.00001 DELTA=1.                   )
      PRINT  80, EVENT, B, D100001, E100001
```

```
      CALL TIME(1HP, 1H1 )
      GO TO 11
 300  CONTINUE
  10  CONTINUE
  11  CONTINUE
  91  FORMAT (10E12.5)
      END
      END
```

REFERENCES

[1] Wald, Abraham. Sequential Analysis, John Wiley & Sons, Inc. New York (1947).

[2] Lehmann, E. L. "Consistency and unbiasedness of certain non-parametric tests," Annals of Mathematical Statistics, Vol. 22, p. 165 (1951).

[3] Chernoff, H. and Teicher, H. "A central limit theorem for sums of interchangeable random variables," Annals of Mathematical Statistics, Vol. 29, pp. 118-130 (1958).

[4] Barnett, V. D. "The behavior of pseudo-random sequences generated on computers by the multiplicative congruential method," Math of Computation, Vol. 16, pp. 63-69 (1962).

[5] Blumenthal, S. "Contributions to the theory of the two-sample problem," Unpublished doctoral thesis, Cornell University (1961).